Scattering theory for Schrödinger equations on manifolds with asymptotically polynomially growing ends

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Abstract

We study a time-dependent scattering theory for Schrödinger operators on a manifold with asymptotically polynomially growing ends. We use the Mourre theory to show the spectral properties of self-adjoint second-order elliptic operators. We prove the existence and the asymptotic completeness of wave operators using the smooth perturbation theory of Kato. We also consider a two-space scattering with a simple reference system.

1 Introduction

We study a class of self-adjoint second-order elliptic operators, which includes Laplacians with long-range potentials on non-compact manifolds which are asymptotically polynomially growing at infinity. We prove Mourre estimate and apply the Mourre theory to these operators. Then we show that there are no accumulation points of embedded eigenvalues except for the zero energy. We obtain resolvent estimates which imply the absence of singular spectrum. We also show the Kato-smoothness for three types of operators. We construct a time-dependent scattering theory for two operators in our class. If the perturbation is "short-range", it admits a factorization into a product of Kato-smooth operators. By virtue of the smooth perturbation theory of Kato, we learn the existence and the asymptotic completeness of wave operators. Lastly, we consider a two-space scattering with a simple reference system. We follow the settings by Ito and Nakamura.

We now describe our model. Let M be an n-dimensional smooth non-compact manifold such that $M = M_C \cup M_\infty$, where M_C is pre-compact and M_∞ is the non-compact end as follows: We assume that M_∞ has the form $\mathbb{R}_+ \times N$ where N is a $n-1$-dimensional compact manifold, and $\mathbb{R}_+ = (0,\infty)$ is the real half line. Let ω be a positive C^∞ density ω on M such that on M_∞,

$$\omega = dr \cdot \mu$$

where r is a coordinate in $\mathbb{R}_+$ and μ is a smooth positive density on N. We set $\mathcal{H} = L^2(M,\omega)$ be our function space. We set our "free operator" a self-adjoint second-order elliptic operator L_0 which has the form:

$$L_0 = D_r^2 + k(r)P \qquad \text{on } (1,\infty)\times N.$$

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Here $D_r = i^{-1}\partial_r$, P is a positive self-adjoint second-order elliptic operator acting on $L^2(N,\mu)$, and k is a positive smooth function of r such that the derivatives of k satisfy the following estimates for some $c_0, C > 0$,

$$\begin{aligned} c_0 r^{-1}k \leq -k' \leq Cr^{-1}k, \\ |k''| \leq Cr^{-2}k. \end{aligned} \tag{1}$$

For example $k(r) = r^{-\alpha}$, with $\alpha > 0$ satisfies the above conditions.

We assume that L is a second-order elliptic operator on M, essentially self-adjoint on $C_0^\infty(M)$, such that

$$L = L_0 + E,$$

with E having the following properties: There are finitely many coordinate charts $(r, \theta_1, \cdots, \theta_{n-1})$ on M_∞ such that in each chart E has the form

$$E = (1, D_r, \sqrt{k}\tilde{D}_\theta) \begin{pmatrix} V & b_1 & b_2 \\ b_1 & a_1 & a_2 \\ {}^t b_2 & {}^t a_2 & a_3 \end{pmatrix} \begin{pmatrix} 1 \\ D_r \\ \sqrt{k}\tilde{D}_\theta \end{pmatrix} \tag{2}$$

where $\mu(\theta)$ is defined by $\mu = \mu(\theta)d_{\theta_1}\cdots d_{\theta_{n-1}}$ and $\tilde{D}_\theta = \mu(\theta)^{-\frac{1}{2}}D_\theta\mu(\theta)^{\frac{1}{2}}$ is self-adjoint on $L^2(N,\mu)$. The coefficients a_1, a_2, b_1, b_2, and V have support in M_∞ and are smooth real-valued functions of $(r, \theta_1, \cdots, \theta_{n-1})$ such that

$$\begin{aligned} |\partial_r^l \partial_\theta^\alpha a_j(r,\theta)| \leq C_{l,\alpha} r^{-\nu_{a_j}-l}, \\ |\partial_r^l \partial_\theta^\alpha b_j(r,\theta)| \leq C_{l,\alpha} r^{-\nu_{b_j}-l}, \\ |\partial_r^l \partial_\theta^\alpha V(r,\theta)| \leq C_{l,\alpha} r^{-\nu_V-l}. \end{aligned} \tag{3}$$

Let $\chi(r) \in C^\infty(\mathbb{R})$ be a $\mathbb{R}_+$-valued function such that $\chi(r) = 1$ if $r \geq 1$ and $\chi(r) = 0$ if $r \leq \frac{1}{2}$, and set $\chi_R(r) = \chi(\frac{r}{R})$ with $R > 0$. We set our dilation generator by:

$$A = \frac{1}{2}(\chi_R^2 r D_r + D_r r \chi_R^2). \tag{4}$$

Now we state the main results.

Theorem 1. *Suppose $L = L_0 + E$, where k satisfies* (1) *and the coefficients in E obey the bounds* (3) *with $\nu = \min\{\nu_{a_i}, \nu_{b_j}, \nu_V\} > 0$. Then $\sigma_{ess}(L) = \mathbb{R}_+ \cup \{0\}$ and L satisfies a Mourre estimate at each point in $\mathbb{R}_+$ with conjugate operator A in the sense of Definition 9. In particular, eigenvalues of L do not accumulate in $\mathbb{R}_+$, and $\sigma_{sc}(L) = \emptyset$. We also obtain the resolvent estimates:*

$$\sup_{z\in\Lambda_\pm=\Lambda\pm i\mathbb{R}_+} \|(|A|+1)^{-s}(L-z)^{-1}(|A|+1)^{-s}\| < \infty$$

if $\Lambda \Subset \mathbb{R}\setminus\sigma_{pp}(L)$ and $s > \frac{1}{2}$.

We prove Theorem 1 in Section 2.

Theorem 2. *Under the hypotheses of Theorem 1, the operators*

$$\begin{aligned} G_0 &= \langle r\rangle^{-s}, \\ G_1 &= \chi_R \langle r\rangle^{-s} D_r, \\ G_2 &= \chi_R \langle r\rangle^{-\frac{1}{2}} (kP)^{\frac{1}{2}} \end{aligned}$$

are L-smooth on Λ if $\Lambda \Subset \mathbb{R}\setminus\sigma_{pp}(L)$ and $s > \frac{1}{2}$.

We prove Theorem 2 in Section 3 and Section 4.

Theorem 3. *Suppose the short-range condition for E, that is, $\nu_{a_1} = \nu_{a_2} = \nu_{b_1} = \nu_{b_1} = \nu_V > 1$, and $\nu_{a_3} = 1$. Then the wave operators*

$$W^{\pm}(L, L_0) := s\text{-}\lim_{t\to\pm\infty} e^{itL}e^{-itL_0}P_{ac}(L_0)$$

and $W^{\pm}(L_0, L)$ exist and are adjoint each other. They are complete and give the unitarily equivalence between $L_0^{(ac)}$ and $L^{(ac)}$.

We prove Theorem 3 in Section 5. We note that the wave operators $W^{\pm}(L_2, L_1)$ exist and are asymptotically complete if both of L_1 and L_2 satisfy the hypotheses of Theorem 1 (long-range) but the difference $L_2 - L_1$ is short-range in the sense of Theorem 3.

Next we consider a two-space scattering. We prepare a reference system as follows:

$$M_f = \mathbb{R}\times N, \quad \mathcal{H}_f = L^2(M_f, H(\theta)drd\theta),$$
$$H_0 = D_r^2 \text{ on } M_f,$$
$$H_k = D_r^2 + k(r)P \text{ on } M_f.$$

Note that H_0 and H_k are essentially self-adjoint on $C_0^\infty(M_f)$, and we denote the unique self-adjoint extensions by the same symbols. We define the identification operator $J : \mathcal{H}_f \to \mathcal{H}$ by

$$(Ju)(r,\theta) = \chi(r)u(r,\theta) \text{ if } (r,\theta)\in M_\infty$$

and $Ju(x) = 0$ if $x \notin M_\infty$, where $u \in \mathcal{H}_f$. We denote the Fourier transform with respect to r variable by $\mathcal{F}$:

$$(\mathcal{F}u)(\rho,\theta) = \frac{1}{\sqrt{2\pi}}\int e^{ir\rho}u(r,\theta)dr.$$

We set

$$\mathcal{H}_f^{\pm} := \mathcal{F}^{-1}[1_{\mathbb{R}_\pm}(\rho)L^2(\mathbb{R}\times N : d\rho\cdot\mu)].$$

Then $\mathcal{H}_f = \mathcal{H}_f^+ \oplus \mathcal{H}_f^-$.

In the two-space scattering, we need additional conditions on k:

Definition 4. *Suppose that k is a positive smooth function of r satisfying* (1). *k is said to be short-range if*

$$|k(r)| \le C\langle r\rangle^{-\nu_k} \tag{5}$$

with $\nu_k > 1$. k is said to be smooth long-range if

$$|\partial_r^l k(r)| \le C\langle r\rangle^{-\nu_k - l} \tag{6}$$

with $l\in\mathbb{N}$, and $\nu_k > 0$.

For short-range k, we have the following.

Theorem 5. *Suppose the hypotheses of Theorem 3 and that k is short-range. Then the wave operators $W^{\pm}(L, H_0; J)$ and $W^{\pm}(H_0, L; J^*)$ exist and are adjoint each other. The asymptotic completeness*

$$W^{\pm}(L, H_0; J)\mathcal{H}_f^{\pm} = P_{ac}(L)\mathcal{H}$$

holds.

For long-range k, we need to modify the identifier.

Theorem 6. *Suppose k is smooth long-range in the sense of Definition 4. Fix $\Lambda \Subset \mathbb{R}$. Then there exists suitable operators $J^{\pm} \in B(\mathcal{H}_f)$ such that the wave operators $W^{\pm}(L, H_0; JJ^{\pm})$ and $W^{\pm}(H_0, L; (JJ^{\pm})^*)$ exist and are isometric on $E_{\Lambda}(H_0)\mathcal{H}_f^{\pm}$ and $E_{\Lambda}(L)P_{ac}(L)\mathcal{H}$, respectively, $W^{\pm}(L, H_0; JJ^{\pm})\mathcal{H}_f^{\mp} = 0$, and the asymptotic completeness*

$$W^{\pm}(L, H_0; JJ^{\pm})E_{\Lambda}(H_0)\mathcal{H}_f^{\pm} = E_{\Lambda}(L)P_{ac}(L)\mathcal{H}$$

holds.

The construction of modifiers $J^{\pm}$ will be given in Section 6. We can also admit a_1 to have a long-range part which depends only on r. For details, see Section 6.

There is a long history on spectral and scattering theory for Schrödinger operators (see, for example, [15], [19] and references therein). Much of works are connected to differential operators on a Euclidean space. The spectral properties of Laplace operators on a class of non-compact manifolds were studied by Froese, Hislop and Perry [4, 5], and Donnelly [3] using the Mourre theory (see, the original paper Mourre [12], and Perry, Sigal, and Simon[13]). We follow the settings in Froese and Hislop [4], and Theorem 1 may be seen as a direct generalization of [4]. We note that only the case with $\nu = 1$ is treated in [4].

In early 1990s, Melrose introduced a new framework of scattering theory on a class of Riemannian manifolds with metrics called scattering metrics (see [11] and references therein). He and the other authors have studied Laplace operators on such manifolds. They also studied the absolute scattering matrix, which is defined through the asymptotic expansion of generalized eigenfunctions.

Debièvre, Hislop, and Sigal [2] studied a time-dependent scattering theory and proved its asymptotic completeness for some classes of manifolds, including manifolds with asymptotically growing ends with $\nu > 1$.

Ito and Nakamura [7] studied a time-dependent scattering theory for Schrödinger operators on scattering manifolds. They used the two-space scattering framework of Kato [9] with a simple reference operator D_r^2 on a space of the form $\mathbb{R} \times N$, where N is the boundary of the scattering manifold M.

The case where $M = M_C \cup M_{\infty}$ is a Riemannian manifold, the metric on M_{∞} is "close" to a warped product of $\mathbb{R}_+$ and a compact manifold N, and L is the Laplace operator, fits into our framework. The function $\sqrt{k(r)}$, varies inversely with the size of $M_{\infty} = \mathbb{R}_+ \times N$. A typical exapmle of k which satisfies (1) is given by $k(r) = r^{-\alpha}, \alpha > 0$. The case $\alpha = 2$ corresponds to scattering manifolds including asymptotically Euclidean spaces. By using results of Ito and Nakamura [7] twice, and by applying the chain rule for wave operators, we can show the existence and the asymptotic completeness of wave operators on scattering manifolds in the one-space scattering framework. Therefore our results can be considered as a generalization of [7] for all $\alpha > 0$. In [7], assumptions on a_2 and a_3 are weakend to long-range perturbations.

Our proof of the existence and the asymptotic completeness of wave operators depends on the smooth perturbation theory of Kato [8] (see also Yafaev [17] and [19]). The Kato smoothness of $G_0 = \langle r \rangle^{-s}$, $s > \frac{1}{2}$ in Theorem 2 is closely related to the limiting absorption principle. The resolvent estimates in the Mourre theory (Theorem 1) imply the limiting absorption principle via a technique in Section 8 of [13]. The Kato smoothness of $G_1 = \chi_R \langle r \rangle^{-s} D_r$ will be obtained in a similar way, but we have to extend the technique in [13] from $\alpha = 1$ to $\alpha = 2$ (Lemma 21 (i)). The Kato-smoothness of $G_2 = \chi_R \langle r \rangle^{-\frac{1}{2}} (kP)^{\frac{1}{2}}$ is called the radiation estimates. Our proof is quite similar to the one [18], which relies on the commutator method (see Putnam [14] and Kato [10]).

The limiting absorption principle suffices to show the asymptotic completeness in the case of two-particle Hamiltonians with short-range scalar potentials. However, radiation estimates

are crucial in scattering for long-range potentials on a Euclidean space (see Yafaev [19]). In this paper, we found that radiation estimates are also useful for handling short-range metric perturbations and magnetic potentials. We hope that we can also construct appropriate modifiers so that the technique of Yafaev can be applied to show the existence and the asymptotic completeness of modified wave operators with long range perturbations in our settings.

In the two-space scattering, essentially we only need to examine wave operators for the pair (H_k, H_0). However, since P commutes with both of H_k and H_0, it reduces to the 1-dimensional scattering. When k is short-range, we only need a narutal identifier. When k is long-range, we will construct a 1-dimensional modifiers for the corresponding 1-dimensional long-range scattering.

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2 Application of Mourre Theory

In this section, we prove Theorem 1. For the sake of completeness, we give a detailed proof. But methods and proofs used here are almost the same as those in [4] and [2], where $\nu = 2$ and $\nu = 1$, respectively, are assumed. We will prove the Mourre estimate under the condition $\nu > 0$. The index ν will explicitly appear, for example, in Lemma 18 and Lemma 13.

We first quote the Mourre theory. We define a scale of spaces associated to a self-adjoint operator L.

Definition 7 (Scale of spaces)**.** *Let L be a self-adjoint operator on a Hilbert space $\mathcal{H}$. For $s \geq 0$ define $\mathcal{H}_s = D((1+|L|)^{\frac{s}{2}})$ with the graph norm*

$$\|\psi\|_s := \|(1+|L|)^{\frac{s}{2}})\psi\|.$$

Define $\mathcal{H}_{-s}$ to be the dual spaces of $\mathcal{H}_s$ thought of as the closure of $\mathcal{H}$ in the norm

$$\|\psi\|_{-s} := \|(1+|L|)^{-\frac{s}{2}})\psi\|.$$

Definition 8 (Conjugate Operators)**.** *Let L be a self-adjoint operator on a Hilbert space $\mathcal{H}$ and $\mathcal{H}_s$ be the scale of spaces associated to L. A self adjoint operator A is called a conjugate operator of L if*

(i). $D(A) \cap \mathcal{H}_2$ is dense in $\mathcal{H}_2$,

(ii). the form $[L, iA]$ defined on $D(A) \cap \mathcal{H}_2$ is bounded below and extends to a bounded operator from $\mathcal{H}_2$ to $\mathcal{H}_{-1}$,

(iii). there is a self-adjoint operator L_0 with $D(L_0) = D(L)$ such that $[L_0, iA]$ extends to a bounded map from $\mathcal{H}_2$ to $\mathcal{H}$, and $D(A) \cap D(L_0 A)$ is a core for L_0,

(iv). the form $[[L, iA], iA]$ extends from $\mathcal{H}_2 \cap D(LA)$ to a bounded operator from $\mathcal{H}_2$ to $\mathcal{H}_{-2}$.

(v). e^{itA} leaves $\mathcal{H}_2$ invariant and for each $\psi \in \mathcal{H}_2$, $\sup_{|t|\leq 1} \|e^{itA}\psi\|_2 < \infty$.

Definition 9 (Mourre Estimate)**.** *A self-adjoint operator L satisfies a Mourre estimate on an interval $\Lambda \subset \mathbb{R}$ with conjugate operator A if A is a conjugate operator of L such that there exist a positive constant α and a compact operator K such that*

$$E_\Lambda [L, iA] E_\Lambda \geq \alpha E_\Lambda + K.$$

Here $E_\Lambda = E_\Lambda(L)$ is the spectral projection for L. We say that L satisfies a Mourre estimate at a point $\lambda \in \mathbb{R}$ if there exists an interval Λ containing λ such that L satisfies a Mourre estimate on Λ.

Now we state the Mourre theory.

Theorem 10 (Mourre)**.** *Suppose that a self-adjoint operator L satisfies a Mourre estimate at $\lambda \in \mathbb{R}$ with a conjugate operator A. Then there exists an open interval Λ containing λ such that L has finitely many eigenvalues in Λ and each eigenvalue has finite multiplicity. If $\lambda \notin \sigma_{pp}(L)$, then there exists an open interval Λ containing λ such that L has no singular continuous spectrum in Λ and for $s > \frac{1}{2}$,*

$$\sup_{z \in \Lambda_\pm = \Lambda \pm i\mathbb{R}_+} \|(|A|+1)^{-s}(L-z)^{-1}(|A|+1)^{-s}\| < \infty.$$

We refer to [12] and [13] for the proof of this theorem.

In the following of this section we will show that the hypotheses of the Theorem 10 will be satisfied for the case stated in section 1.

Lemma 11. *Suppose $f \in C_0^\infty(\mathbb{R})$, D is a differential operator with smooth coefficients, and χ is a smooth cut-off function with compact support. Then $\chi D f(L)$ and $\chi D f(L_0)$ are compact operators from $L^2(M)$ to $L^2(M)$.*

Proof. Let Ω be a bounded domain with smooth boundary which contains supp χ. Then $\chi D f(L_0)$ and $\chi D f(L)$ map $L^2(M)$ to a Sobolev space $H^s(\Omega)$ for any $s > 0$. But $H^s \hookrightarrow L^2(\Omega) \hookrightarrow L^2(M)$, and the first embedding is compact by Rellich's theorem. $\square$

Lemma 12. *Let $\mathcal{H}_s$ be the scale of spaces associated with L_0. Then*

(i). $\chi_R D_r : \mathcal{H}_s \to \mathcal{H}_{s-1}$ is bounded for $s \in [-1, 2]$,

(ii). $\chi_R D_r^2 : \mathcal{H}_s \to \mathcal{H}_{s-2}$ is bounded for $s \in [0, 2]$,

(iii). $\chi_R (kP+1)^{\frac{1}{2}} : \mathcal{H}_s \to \mathcal{H}_{s-1}$ is bounded for $s \in [-1, 2]$,

(iv). $\chi_R (kP+1) : \mathcal{H}_s \to \mathcal{H}_{s-2}$ is bounded for $s \in [0, 2]$.

Proof of Lemma 12. We begin by proving that

$$\|\chi_R D_r (L_0 + C)^{-\frac{1}{2}}\| \leq 1 \tag{7}$$

for some constant C. Choose a positive constant C_1 so that $L_0 + C_1$ is a positive operator. Let $\tilde{\chi}_R = (1-\chi_R)^{-\frac{1}{2}}$. The IMS localozation formula gives

$$L_0 + C_1 = \chi_R (L_0 + C_1)\chi_R + \tilde{\chi}_R (L_0 + C_1)\tilde{\chi}_R - (\chi_R')^2 - (\tilde{\chi}_R')^2.$$

This implies

$$\begin{aligned} L_0 + C_1 &\geq \chi_R D_r^2 \chi_R - (\chi_R')^2 - (\tilde{\chi}_R')^2 \\ &\geq D_r \chi_R^2 D_r - \chi_R'' \chi_R - (\chi_R')^2 - (\tilde{\chi}_R')^2 \end{aligned}$$

as form inequalities on C_0^∞. Since

$$|\chi_R''\chi_R - (\chi_R')^2 - (\tilde{\chi}_R')^2| \leq \frac{C}{R^2},$$

this implies

$$D_r \chi_R^2 D_r \leq L_0 + C.$$

This shows for $\phi \in C_0^\infty$,

$$\|\chi_R D_r \phi\| \leq \|(L_0 + C)^{\frac{1}{2}} \phi\|. \tag{8}$$

Since C_0^∞ is a core for $(L_0 + C)^{\frac{1}{2}}$, we can find for every $\phi \in D((L_0 + C)^{\frac{1}{2}})$, a sequence $\phi_n \in C_0^\infty$ such that $\phi_n \to \phi$ and $(L_0 + C)^{\frac{1}{2}} \phi_n \to (L_0 + C)^{\frac{1}{2}} \phi$. Then

$$\begin{aligned}
|(D_r \chi_R \psi, \phi)| &= \lim_{n\to\infty} |(D_r \chi_R \psi, \phi)| \\
&= \lim_{n\to\infty} |(\psi, \chi_R D_r \phi)| \\
&= \limsup_{n\to\infty} \|\psi\| \|(L_0 + C)^{\frac{1}{2}} \phi_n\| \\
&\leq \|\psi\| \|(L_0 + C)^{\frac{1}{2}} \phi\|
\end{aligned}$$

which shows that $\phi \in D(\chi_R D_r)$ and that (8) holds for any $\phi \in D((L_0 + C)^{\frac{1}{2}})$. Writing $\phi = (L_0 + C)^{-\frac{1}{2}} \psi$ for $\psi \in L^2$, we see that this implies (7).

Next we will prove that

$$\|\chi_R D_r^2 (L_0 + C)^{-1}\| \leq C. \tag{9}$$

With C_1 as above,

$$\begin{aligned}
(L_0 + C_1)^2 &\geq (L_0 + C_1) \chi_R^2 (L_0 + C_1) \\
&= D_r^2 \chi_R^2 D_r^2 + D_r^2 \chi_R^2 (kP + C_1) + (kP + C_1) \chi_R^2 D_r^2 + (kP + C_1)^2 \chi_R^2 \\
&= D_r^2 \chi_R^2 D_r^2 + 2 D_r \chi_R^2 (kP + C_1) D_r - (\chi_R^2 (kP + C_1))'' + (kP + C_1)^2 \chi_R^2.
\end{aligned}$$

Using the fact that $|k'|$ and $|k''|$ are bounded by a constant times k, we see that

$$(\chi_R^2 (kP + C_1))'' \leq C \chi (kP + C_1) \chi$$

for some cut-off function χ. Using the IMS formula again, this implies

$$(\chi_R^2 (kP + C_1))'' \leq C(L_0 + C)$$

for some C. Since $2 D_r \chi_R^2 (kP + C_1) D_r + (kP + C_1)^2 \chi_R^2 \geq 0$, we obtain

$$(L_0 + C_1)^2 \geq D_r^2 \chi_R^2 D_r^2 - C(L_0 + C),$$

which implies for some C,

$$D_r^2 \chi_R^2 D_r^2 \leq C(L_0 + C)^2,$$

which leads to (9) as in the proof of (7).

A similar argument shows that

$$\|D_r^2 \chi_R (L_0 + C)^{-1}\| \leq C. \tag{10}$$

Now by complex interpolation, (9) and (10) implies

$$\|(L_0 + C)^{-1+z} D_r^2 \chi_R (L_0 + C)^{-z}\| \leq C.$$

for Rez in $[0, 1]$, which implies (ii) of Lemma.

To prove (i) using complex interpolation, one need to prove

$$\|(L_0 + C)^{-1} \chi_R D_r (L_0 + C)^{\frac{1}{2}}\| \leq C, \tag{11}$$

$$\|(L_0 + C)^{\frac{1}{2}} \chi_R D_r (L_0 + C)^{-1}\| \leq C. \tag{12}$$

Examining (11), we see that

$$\begin{aligned}&\|(L_0+C)^{-1}\chi_R D_r(L_0+C)^{\frac{1}{2}}\| \\ &\le \|\chi_R D_r(L_0+C)^{\frac{1}{2}}\| + \|(L_0+C)^{-1}[L_0,\chi_R D_r](L_0+C)^{-\frac{1}{2}}\|.\end{aligned}$$

The first term on the right hand side is bounded by (7). The second term can be decomposed into two terms according to the following equation:

$$[L_0,\chi_R D_r] = [D_r^2,\chi_R D_r] + [kP,\chi_R D_r].$$

The first one is bounded using (ii) of Lemma; the second is bounded by

$$[\chi_R, i\chi_R D_r] = \chi_R kP \le C\chi_R kP \le C(L_0+C)$$

and an argument similar to the one above. This gives (11). (12) follows similarly.

(iii) and (iv) follow in a similar way. □

Lemma 13. *Let L, L_0 and E be as in Theorem 1 and let $\mathcal{H}_s$ be the scale of spaces associated with L_0. Then*

(i). $\langle r\rangle^\nu E : \mathcal{H}_s \to \mathcal{H}_{s-2}$ is bounded for $s \in [0,2]$,

(ii). by taking R large enough in the definition of E, we may assume that the relative L_0-bound of E is less than 1,

(iii). $(L_0-z)^{-1} - (L-z)^{-1}$ is compact for $Im(z) \neq 0$.

Proof of Lemma13. (i) will follow by complex interpolation if we can show

$$\begin{aligned}&\|\langle r\rangle^\nu E(L_0+i)^{-1}\| \le C,\\ &\|(L_0+i)^{-1}\langle r\rangle^\nu E\| \le C.\end{aligned}$$

As a typical exapmle, we choose $\sqrt{k}\tilde{D}_\theta a_2 D_r$. Then

$$\begin{aligned}&\|\langle r\rangle^\nu \chi_R\sqrt{k}\tilde{D}_\theta a_2 D_r\chi_R(L_0+i)^{-1}\| \\ &\le \|\langle r\rangle^\nu\sqrt{k}a_2\tilde{D}_\theta\chi_R D_r(L_0+i)^{-1}\| + \|\langle r\rangle^\nu\sqrt{k}(\tilde{D}_\theta a_2)\chi_R D_r(L_0+i)^{-1}\| \\ &\le \sup_r\{\langle r\rangle^\nu|a_2|\}\cdot\|\sqrt{k}\tilde{D}_\theta(kP+1)^{-\frac{1}{2}}\|\cdot\|\chi_R(kP+1)^{\frac{1}{2}}D_r(L_0+i)^{-1}\| \\ &+\sup_r\{\langle r\rangle^\nu|\tilde{D}_\theta a_2|\}\cdot\|\sqrt{k}\chi_R D_r(L_0+i)^{-1}\| \\ &\le C,\end{aligned}$$

by assumptions on a_2 and Lemma 12.

To prove (iii), we use the resolvent formula

$$\begin{aligned}&(L_0-z)^{-1} - (L-z)^{-1} \\ &= (L-z)^{-1}E(L_0-z)^{-1} \\ &= (L-z)^{-1}\langle r\rangle^{-\nu}\chi_R\langle r\rangle^\nu E(L_0-z)^{-1}.\end{aligned}$$

The operator $(L-z)^{-1}\langle r\rangle^{-\nu}\chi_R$ can be approximated in norm by operators $f(L)\chi$ considerd in Lemma 11, and thus is compact while $\langle r\rangle^\nu E(L_0-z)^{-1}$ is bounded by (i). This proves (iii). □

Lemma 14. *$D(L) = D(L_0)$ and the scale of spaces associated to L and L_0 are the same. If $f \in C_0^\infty$, then $f(L) - f(L_0)$ is compact.*

Proof of Lemma 14. The first statement is obtained by the relative boundedness, and the second follows from (iii) of Lemma 13 and a Stone-Weierstrass argument. ☐

Lemma 15. $\sigma_{ess}(L) = [0, \infty)$.

Proof. The Persson's formula (see, for exapmle, [1])

$$\inf \sigma_{\mathrm{ess}}(L) = \sup_{K \Subset M} \inf_{\phi \in C_0^\infty(M \setminus K), \|\phi\|=1} \langle \phi, L\phi \rangle$$

and a Weyl sequene argument give the desired result. ☐

Lemma 16. *Let L_0 be as in Theorem 1. Then for large enough R,*

(i). $[L_0, iA]$ extends from C_0^∞ to a bounded operator $\mathcal{H}_{+2} \to \mathcal{H}$,

(ii). $[[L_0, iA], iA]$ extends from C_0^∞ to a bounded operator $\mathcal{H}_{+2} \to \mathcal{H}_{-2}$.

Proof. To begin, we show that $[D_r^2, iA]$ is bounded from $\mathcal{H}_{+2}$ to $\mathcal{H}$. A brief calculation shows

$$[D_r^2, iA] = 2(\chi_R^2 r)' D_r^2 + \frac{2}{i}(\chi_R^2 r)'' D_r - \frac{1}{2}(\chi_R^2 r)'''.$$

The coefficients $(\chi_R^2 r)'$, $(\chi_R^2 r)''$, and $(\chi_R^2 r)'''$ are bounded. By taking R large enough, the boundedness of $[D_r^2, iA]$ from $\mathcal{H}_{+2}$ to $\mathcal{H}$ follows from that of $\chi_R D_r^2$ and $\chi_R D_r$, which is ensured by Lemma 12.

Next we consider the term

$$[kP, iA] = -\chi_R^2 r k' P.$$

By (1), $|rk'| \leq Ck$. Using Lemma 12, it follows that $[kP, iA]$ is bounded from $\mathcal{H}_{+2}$ to $\mathcal{H}$. This completes the proof of (i).

The boundedness of the double commutator in (ii) is proven using similar arguments. We can use Lemma 12 to prove the boundedness of $[[D_r^2, iA], iA]$ from $\mathcal{H}_{+2}$ to $\mathcal{H}_{-2}$. Since

$$[[kP, iA], iA] = \chi_R^2 r (\chi_R^2 r k')' P,$$

we need the estimates (1) on the second derivative of k for r large

$$|r^2 k''| \leq Ck$$

to prove the boundedness of $[[kP, iA], iA]$. ☐

Now we prove the Mourre estimate for unperturbated system L_0.

Lemma 17. *Let L_0 be as in Theorem 1 and A given by (4). Suppose $\lambda_0 > 0$. Then for every $\epsilon > 0$ there exist an interval Λ about λ_0 and a compact operator K such that for R large,*

$$E_\Lambda(L_0)[L_0, iA]E_\Lambda(L_0) \geq \min(2, c_0)(\lambda_0 - \epsilon)E_\Lambda(L_0) + K.$$

Here $E_\Lambda(L_0)$ is the spectral projection for L_0 corresponding to Λ, and c_0 is the constant which appears in (1).

Proof. Choosing R large, we have

$$\begin{aligned}[D_r^2, iA] &= 2D_r(\chi_R^2 r)' D_r - \frac{1}{2}(\chi_R^2 r)''' \\ &\geq 2D_r \chi_R^2 D_r - \frac{\epsilon}{4}\min\{2, c_0\} \\ &\geq 2\chi_R D_r^2 \chi_R - \frac{\epsilon}{2}\min\{2, c_0\}.\end{aligned}$$

Also,

$$[kP, iA] = -\chi_R^2 r k' P \geq c_0 \chi_R^2 kP.$$

Combining these two inequalities, we obtain

$$\begin{aligned}[L_0, iA] &\geq \chi_R(2D_r^2 + c_0 kP)\chi_R - \frac{\epsilon}{2}\min\{2, c_0\} \\ &\geq \min\{2, c_0\}(\chi_R L_0 \chi_R - \frac{\epsilon}{2}).\end{aligned}$$

We now multiply this estimate on both sides with $f(L_0)$ where f is a smooth compactly supported characteristic function of an interval about λ_0. This gives

$$f(L_0)[L_0, iA]f(L_0) \geq \min\{2, c_0\}(f(L_0)\chi_R L_0 \chi_R f(L_0) - \frac{\epsilon}{2} f^2(L_0)).$$

Now

$$\begin{aligned}&f(L_0)\chi_R L_0 \chi_R f(L_0) \\ =&f(L_0)L_0(\chi_R - 1)f(L_0) + f(L_0)(\chi_R - 1)L_0\chi_R f(L_0) + f(L_0)L_0 f(L_0).\end{aligned}$$

$f \in C_0^\infty$ implies $f(L_0)L_0$ is bounded. It is not difficult to see that $L_0\chi_R f(L_0)$ is bounded using Lemma 12. Since $\chi_R - 1$ has compact support, $(\chi_R - 1)f(L_0)$ is compact by Lemma 11. Thus, if the support of f is within $\frac{\epsilon}{2}$ of λ_0, we have

$$f(L_0)\chi_R L_0 \chi_R f(L_0) \geq (\lambda_0 - \frac{\epsilon}{2})f^2(L_0) + K.$$

where K is a compact operator. Therefore we have

$$f(L_0)[L_0, iA]f(L_0) \geq \min\{2, c_0\}(\lambda_0 - \epsilon)f^2(L_0) + K. \tag{13}$$

Taking $f = 1$ in a neighbourhood of λ_0 and multiplying from both sides with $E_\Lambda(L_0)$, with Λ small enough to ensure $E_\Lambda(L_0)f(L_0) = E_\Lambda(L_0)$, this inequality gives the desired Mourre estimate.

□

Lemma 18. *Under the hypotheses of Theorem 1,*

(i). $[E, iA] : \mathcal{H}_{+2} \to \mathcal{H}_0$ *is bounded,*

(ii). $f(L)[E, iA]f(L)$*: is compact for* $f \in C_0^\infty$*,*

(iii). $[[E, iA], iA] : \mathcal{H}_{+2} \to \mathcal{H}_{-2}$ *is bounded.*

Proof. It is easy to see that $[E, iA]$ has the following form:

$$[E, iA] = (1, D_r, \sqrt{k}\tilde{D}_\theta)\chi_R \begin{pmatrix} \tilde{V} & \tilde{b}_1 & \tilde{b}_2 \\ \tilde{b}_1 & \tilde{a}_1 & \tilde{a}_2 \\ {}^t\tilde{b}_2 & {}^t\tilde{a}_2 & \tilde{a}_3 \end{pmatrix} \chi_R \begin{pmatrix} 1 \\ D_r \\ \sqrt{k}\tilde{D}_\theta \end{pmatrix} \tag{14}$$

where $\tilde{e} = \tilde{a}_1, \tilde{a}_2, \tilde{b}_1, \tilde{b}_1$ and $\tilde{V}$ satisfy

$$|\tilde{e}(r, \theta)| \leq Cr^{-\nu}, \quad \nu > 0. \tag{15}$$

Here we used the estimates (3) on the first derivatives with respect to r of coefficients in E and the estimates (1) on the first derivative of k. By Lemma 12, (14) and (15) imply $[E, iA]$ is bounded from $\mathcal{H}_{+2} \to \mathcal{H}_0$, which is (i).

The boundedness of the double commutator in (iii) is proven using similar arguments. We need the estimates on second derivatives with respect to r of coefficients in E and k.

To prove (ii), note that $\chi_R(1, D_r, \sqrt{k}\tilde{D}_\theta)f(L)$ is bounded and $\chi_R r^{-\nu}(1, D_r, \sqrt{k}\tilde{D}_\theta)f(L)$ is compact by Lemma 11. Hence (14) implies (ii). □

Proof of Theorem 1. We first show that L and A satisfy the conditions in Definition 8, that is, A is a conjugate operator of L. Since $C_0^\infty \subset D(A) \cap \mathcal{H}_2$ is a core for L by hypothesis, condition (i) in Definition 8 is satisfied. Condition (ii) follows from (i) of Lemma 16 and (i) of Lemma 18. The first statement of (iii) follows from Lemma 14 and (i) of Lemma 16. The second statement follows from the inclusion $C_0^\infty \subset D(A) \cap D(L_0 A)$. Condition (iv) follows from (ii) of Lemma 16 and (iii) of Lemma 18. Let X be a vector field on M such that

$$X = r\chi_R^2(r)\frac{\partial}{\partial r} \quad \text{on} M_\infty,$$

and $X = 0$ on M_C. Let $\{\exp[tX] | t \in \mathbb{R}\}$ be the flow generated by X. The flow induces a one-parameter unitary group defined by

$$U(t)\phi(x) = \Phi(t, x)\phi(\exp[-tX]x)$$

for $\phi \in \mathcal{H}$, where $\Phi(t, x)$ is a weight function to make the dilation operator $U(t)$ unitary. By simple calculation, we find that

$$A = \frac{1}{2}(\chi_R^2 r D_r + D_r r \chi_R^2)$$

is the generator of the dilation operator $U(t)$, that is, $U(t) = e^{-itA}$. Now it is easy to see e^{-itA} leaves $D(L) = \mathcal{H}_2$ invariant and to show (v) as in the Euclidean case.

Now we show the Mourre estimate. We replce L_0 with L in (13). By Lemma 14, $f^2(L) - f^2(L_0)$ and $f(L) - f(L_0)$ are compact. By Lemma 16, we can see that $[L_0, iA]f(L)$ and $f(L_0)[L_0, iA]$ are bounded. $f(L)[E, iA]f(L)$ is compact by (ii) of Lemma 18. Using these facts, it is easily seen that replacing L_0 with L in (13) introduces a compact error, which can be handled in K. Making this replacement and multiplying the resulting equation from both sides with $E_\Lambda(L_0)$, with Λ small enough to ensure $E_\Lambda(L_0)f(L_0) = E_\Lambda(L_0)$, give the Mourre estimate

$$E_\Lambda(L)[L, iA]E_\Lambda(L) \geq \min(2, c_0)(\lambda_0 - \epsilon)E_\Lambda(L) + K.$$

We have showed that L satisfies a Mourre estimate at any point $\lambda_0 > 0$ with conjugate operator A, which completes the proof of Theorem 1. □

3 Limiting Absorption Principle

In this section we show the limiting absorption principle, which leads to the Kato-smoothness of G_0 and G_1 in Theorem 2. We extend the discussion in [13].

We will prove

Theorem 19. *Let L be as in Theorem 1, $s > \frac{1}{2}$, and $\Lambda \Subset \mathbb{R}_+ \setminus \sigma_{pp}(L)$. Then*

$$\sup_{z\in\Lambda_\pm=\Lambda\pm i\mathbb{R}_+} \|(|r|+1)^{-s}(L-z)^{-1}(|r|+1)^{-s}\| < \infty$$
$$\sup_{z\in\Lambda_\pm=\Lambda\pm i\mathbb{R}_+} \|(|r|+1)^{-1-s}A(L-z)^{-1}A(|r|+1)^{-1-s}\| < \infty.$$

The first estimate is called the limiting absorption principle.

As a preliminary we prove

Lemma 20. *Let L be as in Theorem 1. Then*

(i). $[E, ir\chi_R](L+i)^{-1}$ is bounded,

(ii). $[[E, ir\chi_R], ir\chi_R](L+i)^{-1}$ is bounded,

(iii). $\chi_R D_r r\chi_R(L+i)^{-1}\langle r\rangle^{-1}$ is bounded,

(iv). $\chi_R D_r^2 r^2\chi_R(L+i)^{-1}\langle r\rangle^{-2}$ is bounded.

Proof. By (1), (3) and Lemma 12, we can show (i) and (ii).

Now we compute (iii).

$$\begin{aligned}&\chi_R D_r r\chi_R(L+i)^{-1}\langle r\rangle^{-1}\\ =&\chi_R D_r(L+i)^{-1}r\chi_R\langle r\rangle^{-1} + \chi_R D_r(L+i)^{-1}[L, r\chi_R](L+i)^{-1}\langle r\rangle^{-1}\end{aligned}$$

The first term in the right hand side is bounded by (i) of Lemma 12. We have

$$\begin{aligned}[L, r\chi_R] &= [D_r^2, r\chi_R] + [E, r\chi_R]\\ &= 2i^{-1}(r\chi_R)'D_r - (r\chi_R)'' + [E, r\chi_R].\end{aligned}$$

Using (i) of Lemma 12 and (i) of Lemma 20, we obtain the boundedness of $[L, r\chi_R](L+i)^{-1}$, which implies the boundedness of the second term.

Next we will show (iv). we begin with the equality

$$\chi_R D_r^2 r^2\chi_R(L+i)^{-1}\langle r\rangle^{-2} = \chi_R D_r^2(L+i)^{-1}r^2\chi_R\langle r\rangle^{-2} + \chi_R D_r^2(L+i)^{-1}[L, r^2\chi_R](L+i)^{-1}\langle r\rangle^{-2}.$$

The first term in the right hand side is bounded by (ii) of Lemma 12. The second term can be decomposed into two terms according to the following:

$$[L, r^2\chi_R] = [D_r^2, r^2\chi_R] + [E, r^2\chi_R].$$

It is easy to see that the first one is bounded using (iii). Replacing χ_R by χ_R^2, the second can be decomposed in the following way:

$$\begin{aligned}&[E, r^2\chi_R^2](L+i)^{-1}\langle r\rangle^{-1}\\ =&2[E, r\chi_R]r\chi_R(L+i)^{-1}\langle r\rangle^{-1} + [r\chi_R, [E, r\chi_R]](L+i)^{-1}\langle r\rangle^{-1}\\ =&2[E, r\chi_R](L+i)^{-1}r\chi_R\langle r\rangle^{-1} + 2[E, r\chi_R](L+i)^{-1}[L, r\chi_R](L+i)^{-1}\langle r\rangle^{-1}\\ &+[r\chi_R, [E, r\chi_R]](L+i)^{-1}\langle r\rangle^{-1},\end{aligned}$$

which is bounded using (i), (ii) and the boundedness of $[D_r^2, r\chi_R](L+i)^{-1}\langle r\rangle^{-1}$, which can be shown by the argument in (iii). This proves (iv). $\square$

Lemma 21. *Let L be as in Theorem 1. Then*

(i). $\langle |A|\rangle^{\alpha}(L+i)^{-1}\langle r\rangle^{-\alpha}$ *is bounded for* $0 \le \alpha \le 2$

(ii). $\langle |A|\rangle^{s}[A,(L+i)^{-1}]\langle r\rangle^{-1-s}$ *is bounded for* $0 \le s \le 1$.

Proof of Lemma 21. By interpolation, it is enough to prove for $\alpha = 0, 2$ and $s = 0, 1$. The case $\alpha = 0$ is obvious. The case $\alpha = 2$ and $s = 1$ follows from (iii) and (iv) of Lemma 20.

The case $s = 0$ follows from Lemma 16 and Lemma 18. □

Proof of Theorem 19. Writing

$$\begin{aligned}&\langle r\rangle^{-s}(L+i)^{-1}(L-z)^{-1}(L+i)^{-1}\langle r\rangle^{-s}\\ =\ &\langle r\rangle^{-s}(L+i)^{-1}\langle |A|\rangle^{-s}\cdot\langle |A|\rangle^{s}(L-z)^{-1}\langle |A|\rangle^{s}\cdot\langle |A|\rangle^{-s}(L+i)^{-1}\langle r\rangle^{-s}\end{aligned}$$

and using Theorem 1 and Lemma 21, we see that

$$\langle r\rangle^{-s}(L+i)^{-1}(L-z)^{-1}(L+i)^{-1}\langle r\rangle^{-s}$$

is bounded.

Also, writing

$$\begin{aligned}&\langle r\rangle^{-1-s}A(L+i)^{-1}(L-z)^{-1}(L+i)^{-1}A\langle r\rangle^{-1-s}\\ =\ &\langle r\rangle^{-1-s}A(L+i)^{-1}\langle |A|\rangle^{-s}\cdot\langle |A|\rangle^{s}(L-z)^{-1}\langle |A|\rangle^{s}\cdot\langle |A|\rangle^{-s}(L+i)^{-1}A\langle r\rangle^{-1-s}\end{aligned}$$

and using Theorem 1 and Lemma 21, we see that

$$\langle r\rangle^{-1-s}A(L+i)^{-1}(L-z)^{-1}(L+i)^{-1}A\langle r\rangle^{-1-s}$$

is bounded.

Since

$$(L-z)^{-1} = (L+i)^{-1} + (z+i)(L+i)^{-2} + (z+i)^2(L+i)^{-1}(L-z)^{-1}(L+i)^{-1},$$

we obtain the desired result. □

4 Radiation Estimates

In this section, we prove the radiation estimates. We want first to recall general definitions of Kato-smoothness and the commutator method which allow us to find new Kato-smooth operators K given Kato-smooth operators G. For details, we refer the textbooks Yafaev [17] and [19].

Definition 22. *An H-bounded operator G is called H-smooth in the sense of Kato if*

$$\begin{aligned}&\sup_{f\in D(H),\|f\|=1}\int_{-\infty}^{\infty}\|Ge^{-iHt}f\|^2dt\\ =&\sup_{z\in\mathbb{R}+i\mathbb{R}}\|G((H-z)^{-1}-(H-\bar{z})^{-1})G^*\|\\ <&\infty.\end{aligned}$$

An operator G is called H-smooth on a Borel set Λ if $GE(\Lambda)$ is H-smooth, which is equivalent to the condition

$$\sup_{z\in\Lambda+i\mathbb{R}}\|G((H-z)^{-1}-(H-\bar{z})^{-1})G^*\| < \infty,$$

where $E(\Lambda)$ is the spectral projection of H on Λ.

Proposition 23. *Suppose that*

$$G^*G \leq i[H, M] + K^*K,$$

where M is a H-bounded operator and K is H-smooth on a Borel set Λ. Then G is also H-smooth on Λ.

For the proof of Proposition 23, see Proposition 1.19 in [19].

Now we return to our problem.

Theorem 24. *Let L be as in Theorem 1. Then for large enough R,*

$$\chi_R r^{-\frac{1}{2}}(kP)^{\frac{1}{2}}$$

is L-smooth on Λ if $\Lambda \Subset \mathbb{R} \setminus \sigma_{pp}(L)$.

We prepare the following lemma.

Lemma 25. *For every $\epsilon > 0$, there exist a constant $C > 0$ such that*

$$(c_0 - \epsilon)G_2^*G_2 \leq [L, iM] + C \sum_{j,k=0,1} G_j^*G_j \tag{16}$$

where

$$\begin{gathered} M = \frac{1}{2}(\chi_R D_r + D_r\chi_R) \\ G_0 = \langle r\rangle^{-s}, \\ G_1 = \chi_R\langle r\rangle^{-s}D_r, \\ G_2 = \chi_R\langle r\rangle^{-\frac{1}{2}}(kP)^{\frac{1}{2}} \\ s = \frac{1}{2}(1+\nu) > \frac{1}{2} \end{gathered}$$

and c_0 is the constant which appears in (1).

Proof of Lemma 25. To calculate the commutator $[L, iM]$, we first remark that

$$[D_r^2, iM] = 2D_r\chi_R' D_r \tag{17}$$

$$[k(r)P, iM] = -\chi_R k' P \geq c_0\chi_R r^{-1}kP. \tag{18}$$

Here we used the inequality (1).

For the perturbation term $[E, iM]$. we can prove

$$|([E, iM]u, u)| \leq C\|G_0u\|^2 + \|G_1u\| + C\|\chi_R r^{-\nu}\|\|G_2u\|^2. \tag{19}$$

It suffices to prove this estimate for each term of E in the sum (2). First we consider the terms involving V.

$$\begin{gathered} [V, iM] = -\chi_R V', \\ |([V, iM]u, u)| \leq C\|\chi_R r^{-\frac{1+\nu}{2}}u\|^2 \leq C\|G_0u\|^2. \end{gathered}$$

For the a_1 part, we have that

$$\begin{gathered} [D_r a_1 D_r, iM] = -D_r(\chi_R a_1)' D_r, \\ |([D_r a_1 D_r, iM]u, u)| \leq C\|G_1u\|^2. \end{gathered}$$

For the a_3 part, we have that

$$[\tilde{D}_\theta k a_3 \tilde{D}_\theta, iM] = -\tilde{D}_\theta \chi_R (k a_3)' \tilde{D}_\theta$$
$$|([\tilde{D}_\theta k a_3 \tilde{D}_\theta, iM]u, u)| \leq C\|\chi_R r^{-\nu}\| \cdot \|\chi_R r^{-\frac{1}{2}}(kP)^{\frac{1}{2}}u\|^2 = C\|\chi_R r^{-\nu}\|\|G_2 u\|^2.$$

Other terms can be handled in a similar way.

Combinig the inequalities (17), (18) and (19), we arrive at the estimate

$$\begin{aligned}([L, iM]u, u) &\geq c_0\|G_2 u\|^2 - C\|G_0 u\|^2 - C\|G_1 u\|^2 - \|\chi_R r^{-\nu}\|\|G_2 u\|^2 \\ &\geq (c_0 - \epsilon)\|G_2 u\|^2 - C\|G_0 u\|^2 - C\|G_1 u\|^2\end{aligned}$$

for an arbitrary $\epsilon > 0$ by taking $R > 0$ large enough. This gives the desired estimate (16). □

Proof of Theorem 24. Fix $\Lambda \Subset \mathbb{R} \setminus \sigma_{\mathrm{pp}}(L)$ and consider (16). The operators G_0 and G_1 are L-smooth on Λ by Theorem 19 and G_2 is L-bounded. The commutator method Proposition 23 implies that the operator G_2 is also L-smooth on Λ. □

Theorem 19 and Theorem 24 directly mean Theorem2.

5 One-space scattering

We recall the smooth method of Kato which assures the existence of wave operators for perturbations that are smooth locally. For more details, see Corollary 4.5.7. in [17].

Theorem 26. *Suppose that H and H_0 are self-adjoint operators on Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_0$ respectively, $J \in B(\mathcal{H}_0, \mathcal{H})$ is the identifier, and the pertuabation $HJ - JH_0$ admits a factorization*

$$HJ - JH_0 = G^* G_0,$$

where G_0 is H_0-bounded and G is H-bounded. Suppose $\{\Lambda_n\}$ is a set of intervals which exhausts the core of the spectra of the operators H_0 and H up to a set of Lebesgue measure zero. If on each of the intervals Λ_n the operator G_0 is H_0-smooth and G is H-smooth, then the wave operators $W^\pm(H, H_0; J)$ and $W^\pm(H_0, H; J^*)$ exist.

Now we apply Theorem 26 to our model.

Proof of Theorem3. First we note that any first order differential operator with compactly supported smooth coefficient function is L_0- and $L-$ locally smooth. This fact can be easily proved as in Section3.

The perturbation term E admits a factorization of the following form

$$E = \sum_{l,m=0,1,2} G_l^* B_{l,m} G_m + E_C$$

where G_l are L_0-smooth on any $\Lambda \Subset \mathbb{R} \setminus \sigma_{\mathrm{pp}}(L_0)$ and L-smooth on any $\Lambda \Subset \mathbb{R} \setminus \sigma_{\mathrm{pp}}(L)$ and E_C is a second-order differential operator with compactly supported coefficient function. Then the smooth perturbation theory of Kato shows the existence of the wave operators $W^\pm(L, L_0)$ and $W^\pm(L_0, L)$, which proves the Theorem. □

6 Two-space scattering

In this section, we consider a two-space scattering.

First we treat the short-range case.

Proposition 27. *Suppose that k is short-range. Then the wave operators $W^{\pm}(H_k, H_0)$ and $W^{\pm}(H_0, H_k)$ exist and are adjoint each other. They are asymptotically complete:*

$$W^{\pm}(H_k, H_0)\mathcal{H}_f = P_{ac}(H_k)\mathcal{H}.$$

Proof. Let $E_P(\Lambda)$ be the spectral projections of P on Λ with $\Lambda \Subset \mathbb{R}$. We decompose the perturbation term with identifier $E_P(\Lambda)$ as follows:

$$H_k E_P(\Lambda) - E_P(\Lambda) H_0 = \sqrt{k} P E_P(\Lambda) \sqrt{k}.$$

The limiting absorption principle implies that $\sqrt{k}$ is locally H_0- and H_k- smooth. $PE_P(\Lambda)$ is bounded. The smooth perturbation theory of Kato implies that the wave operators $W^{\pm}(H_k, H_0; E_P(\Lambda))$ and $W^{\pm}(H_0, H_k; E_P(\Lambda))$ exist and are adjoint each other.

Since P commutes with H_0 and H_k,

$$W^{\pm}(H_k, H_0; E_P(\Lambda)) = W^{\pm}(H_k, H_0) E_P(\Lambda),$$
$$W^{\pm}(H_0, H_k; E_P(\Lambda)) = W^{\pm}(H_0, H_k) E_P(\Lambda).$$

Hence $W^{\pm}(H_k, H_0)$ and $W^{\pm}(H_0, H_k)$ exist and are adoint each other. $\square$

Proposition 28. *Suppose that k is short-range or long-range. Then the wave operators $W^{\pm}(L_0, H_k; J)$ and $W^{\pm}(H_k, L_0; J^*)$ exist and are adjoint each other.*

Proof. The perturbation $L_0 J - J(D_r^2 + k(r)P)$ can be decomposed into a sum of products of first-order differential operator with smooth compactly supported coefficients. Hence we can apply the smooth method of Kato. $\square$

Now we obtain the following:

Theorem 29. *Suppose that k is short-range. Then the wave operators $W^{\pm}(L_0, H_0; J)$ and $W^{\pm}(H_0, L_0; J^*)$ exist and are adjoint each other. $W^{\pm}(L_0, H_0; J)\mathcal{H}_f^{\mp} = 0$. $W^{\pm}(L_0, H_0; J)$ and $W^{\pm}(H_0, L_0; J^*)$ are isometric on $\mathcal{H}_f^{\pm}$ and $P_{ac}(L_0)\mathcal{H}$, respectively, and the asymptotic completeness*

$$W^{\pm}(L_0, H_0; J)\mathcal{H}_f^{\pm} = P_{ac}(L_0)\mathcal{H}$$

holds.

Proof. It follows from Proposition 27 and Proposition 28 that the wave operators $W^{\pm}(L_0, H_0; J)$ and $W^{\pm}(H_0, L_0; J^*)$ exist and are adjoint each other.

For $u \in \mathcal{H}_f^{\pm}$,

$$\lim_{t \to \pm\infty} \|J e^{-itH_0} u\| = \|u\|,$$
$$\lim_{t \to \mp\infty} \|J e^{-itH_0} u\| = 0.$$

Hence $W^{\pm}(L_0, H_0; J)\mathcal{H}_f^{\mp} = 0$, and $W^{\pm}(L_0, H_0; J)$ is isometric on $\mathcal{H}_f^{\pm}$.

To show the isometricity of $W^{\pm}(H_0, L_0; J^*)$, it is enough to check that

$$\lim_{t \to \pm\infty} \|(1 - \chi) e^{-itL_0} u\| = 0$$

for $u \in P_{ac}(L_0)\mathcal{H}$. This follows from the local L_0-smoothness of $1 - \chi$. $\square$

Combining Theorem 3 and Theorem 29, we obtain Theorem 5 by virtue of the chain rule of wave operators. Conversely, Theorem 29 and Theorem 5 imply Theorem 3. Theorem 5 is essentially solved in [7]. Hence Theorem 3 with $k(r) = r^{-2}$ is essentially solved in [7]. Our result may be considered as an extention of [7].

In the following of this section, we consider smooth long-range k. We also suppose that the coefficient a_1 in E is separated into two parts, long-range θ- independent term and short-range term:

$$a_1 = a_1^L(r) + a_1^S(r,\theta) \tag{20}$$

$$|\partial_r^l a_1^L(r)| \leq C_l \langle r\rangle^{-\nu_{a_1^L}-l}, \nu_{a_1^L} > 0 \tag{21}$$

$$|\partial_r^l \partial_\theta^\alpha a_1^S(r,\theta)| \leq C_{l,\alpha} \langle r\rangle^{-\nu_{a_1^S}-l}, \nu_{a_1^S} > 1. \tag{22}$$

Set

$$H_L = D_r(1 + a_1^L(r))D_r + k(r)P.$$

We formulate a long-range scattering theory for the triplet $(H_L, H_0; J^\pm)$ with modified identifiers $J^\pm \in B(\mathcal{H}_f, \mathcal{H}_f)$. Since P commutes with H_0 and H_L, it is natural to choose $J^\pm$ as

$$J^\pm = \int J_\lambda^\pm dE_P(\lambda) \tag{23}$$

where

$$P = \int \lambda dE_P(\lambda)$$

is the spectral decomposition of P, and $J_\lambda^\pm$ are bounded operators $L^2(\mathbb{R}) \to L^2(\mathbb{R})$. Through this decomposition, the problem reduces to the long-range scattering for the triplet $\left(H_{L,\lambda}, H_{0,\lambda}; J_\lambda^\pm\right)$ on the real line, where $H_{L,\lambda} = D_r(1 + a_1^L)D_r + \lambda k(r)$ and $H_{0,\lambda} = D_r^2$ are self-adjoint operators on $L^2(\mathbb{R})$. We choose $J_\lambda^\pm$ as a pseudo-differential operator with oscillating symbols

$$J_\lambda^\pm = \chi_\lambda^\pm(D_r)J(\Phi_\lambda^\pm, a^\pm) \tag{24}$$

$$J(\Phi_\lambda^\pm, a^\pm)u(r) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{ir\rho + i\Phi_\lambda^\pm(r,\rho)} a^\pm(r,\rho)\hat{u}(\rho)d\rho$$

$$a^\pm(r,\rho) = \eta(r)\psi(\rho^2)\sigma^\pm(r,\rho). \tag{25}$$

Here $\eta \in C^\infty(\mathbb{R})$ such that $\eta(r) = 0$ near $r = 0$ and $\eta(r) = 1$ for large $|r|$, $\psi \in C_0^\infty(\mathbb{R}_+)$, $\chi_\lambda^\pm \in C_0^\infty(\mathbb{R})$ and $\sigma^\pm = 1$ if $\pm r\rho > 0$ and $\sigma^\pm = 0$ if $\pm r\rho \leq 0$. We seach for a PDO $J_\lambda^\pm$ such that the perturbation

$$T_\lambda^\pm = H_{L,\lambda}J_\lambda^\pm - J_\lambda^\pm H_{0,\lambda}$$

admits a factorization into a product of $H_{L,\lambda}$- and $H_{0,\lambda}$- smooth operators.

Roughly speaking, up to compact terms, $T_\lambda^\pm$ is also a PDO with symbol

$$t_\lambda^\pm(r,\rho) = ((1 + a_1^L(r))(D_r + \rho)^2 - \rho^2 + \lambda k(r))e^{i\Phi_\lambda^\pm(r,\rho)}a^\pm(r,\rho).$$

Let us compute

$$\begin{aligned}&e^{-i\Phi_\lambda^\pm}(r,\rho)((1 + a_1^L(r))(D_r + \rho)^2 - \rho^2 + \lambda k(r))e^{i\Phi_\lambda^\pm}(r,\rho)\\ =&(1 + a_1^L(r))(\nabla\Phi_\lambda^\pm + \rho)^2 + \lambda k(r) - \rho^2 - i(1 + a_1^L(r))\Delta\Phi_\lambda^\pm.\end{aligned}$$

We want to find $\Phi^{\pm}_\lambda$ such that

$$(1+a_1^L(r))(\nabla\Phi^{\pm}_\lambda+\rho)^2+\lambda k(r)-\rho^2$$

is "small". In the case $a_1^L=0$, and $\nu_k>\frac{1}{2}$, it is enough to set

$$\Phi^{\pm}_\lambda(r,\rho)=-\frac{1}{2\rho}\int_0^r\lambda k(s)ds.$$

For general a_1^L and $\nu_k>0$, we need to apply the method of succesive approximations and to keep $[\nu_k^{-1}]$ (the largest integer which does not exceed ν_k^{-1}) iterations:

Lemma 30. *Let $a_1^L(r),k(r)\in C^\infty(\mathbb{R})$ satisfy the smooth long-range condition:*

$$\begin{aligned}|\partial_r^l a_1^L(r)|&\le C\langle r\rangle^{-\nu_{a_1^L}-l}\\ |\partial_r^l k(r)|&\le C\langle r\rangle^{-\nu_k-l}\end{aligned}\tag{26}$$

with $l\in\mathbb{N}$, and $\nu=\max\{\nu_{a_1^L},\nu_k\}>0$. We assume that ν^{-1} is not an integer. Let $\Lambda\Subset\mathbb{R}\setminus\{0\}$. Then for large enough R, there exists a C^∞-function $\Phi^\pm(r,\rho)$ defined on $(r,\rho)\in\Gamma^\pm(R,\Lambda)=\{(r,\rho)||r|>R,\rho\in\Lambda,\pm r\rho>0\}$ such that

$$|\partial_r^l\partial_\rho^k\Phi^\pm(r,\rho)|\le C(1+|r|)^{1-\nu-l}\tag{27}$$

$$R[\Phi^\pm]:=(1+a_1^L)|\nabla\Phi^\pm+\rho|^2+k(r)-\rho^2$$

$$|\partial_r^l\partial_\rho^k R[\Phi](r,\rho)|\le C(1+|r|)^{-1-\epsilon-l}$$

where $\nabla=\partial_r$ and $\epsilon=\nu([\nu^{-1}]+1)-1>0$.

Proof. We only consider the case Φ^+ with $\Lambda\subset\mathbb{R}_+$, and abbreviate "+" . Other cases are similar to prove.

We fix $R>0$ large enough such that $|a_1^L(r)|<\frac{1}{2}$ for $|r|>R$. Set

$$\begin{aligned}\Phi^{(0)}(r,\rho)&:=0,\\ \Phi^{(1)}(r,\rho)&:=-\int_R^r\frac{k(s)+a_1^L(s)\rho^2}{2(1+a_1^L(s)\rho)}ds,\\ \Phi^{(N+1)}&:=\Phi^{(N)}+\phi^{(N+1)}\\ \phi^{(N+1)}(r,\rho)&:=-\frac{1}{2\rho}\int_R^r\left(|\nabla\Phi^{(N)})(s,\rho)|^2-|\nabla\Phi^{(N-1)}(s,\rho)|^2\right)ds.\end{aligned}$$

with $N\ge1$.

A simple computation gives

$$\begin{aligned}R[\Phi^{(2)}]&=(1+a_1^L)(|\nabla\Phi^{(2)}|^2-|\nabla\Phi^{(1)}|^2),\\ R[\Phi^{(N+1)}]&=(1+a_1^L)(|\nabla\Phi^{(N+1)}|^2-|\nabla\Phi^{(N)}|^2)+R[\Phi^{(N)}]+2(1+a_1)\langle\nabla\phi^{(N+1)},\rho\rangle.\end{aligned}$$

Hence by inducetion we have

$$R[\Phi^{(N+1)}]=(1+a_1^L)(|\nabla\Phi^{(N+1)}|^2-|\nabla\Phi^{(N)}|^2).$$

We have uniformly for $\rho\in\Lambda$,

$$\begin{aligned}|\partial_r^l\partial_\rho^k\Phi^{(N)}|&\le C(1+|r|)^{1-\nu-l},\\ |\partial_r^l\partial_\rho^k\phi^{(N)}|&\le C(1+|r|)^{1-N\nu-l},\\ |\partial_r^l\partial_\rho^k R[\Phi^{(N)}]|&\le C(1+|r|)^{-(1+N)\nu-l}.\end{aligned}$$

It is now sufficient to set $\Phi=\Phi^{([\nu^{-1}])}$. $\square$

From now on, we assume that $\Phi_\lambda^\pm$ satisfy the conclusions of Lemma 30 with k replaced by λk. We also assume that $\eta(r) = 0$ if $|r| < R$ and $\chi_\lambda^\pm(\rho) = 1$ near $\{\rho + \nabla_r \Phi_\lambda^\pm(r,\rho) : \rho^2 \in \operatorname{supp}\psi, |r| > R\}$. Now we state the existence of modified wave operators:

Lemma 31. *The wave operators*

$$W^\pm(H_L, H_0; J^\pm), W^\pm(H_0, H_L; (J^\pm)^*) \tag{28}$$

and

$$W^\pm(H_L, H_0; J^\mp), W^\pm(H_0, H_L; (J^\mp)^*) \tag{29}$$

exist. Operators (28) *as well as* (29) *are adjoint each other.*

Proof. It is enough to consider the scattering theory for the triplets $(H_{L,\lambda}, H_{0,\lambda}, J_\lambda^\pm)$.

Set $b = (i^{-1}(\partial_r a_1^L)\rho + (1 + a_1^L)\rho^2 + \lambda k(r))\chi_\lambda^\pm(\rho)$. a and b are in $\mathcal{S}^0$. By Theorem 39, there exists $d \in \mathcal{S}^{m_d}$ with $m_d = 0$ such that

$$\begin{aligned} H_{L,\lambda} J_\lambda^\pm &= (D_r(1 + a_1^L)D_r + \lambda k(r))\chi_\lambda^\pm(D_r)J(\Phi_\lambda^\pm, a^\pm) = b(x, D_r)J(\Phi_\lambda^\pm, a^\pm) \\ &= J(\Phi_\lambda^\pm, d) \end{aligned}$$

and admits the asymptotic expansion

$$d = \sum_{l \geq 0} \frac{1}{l!} d_l,$$
$$d_l(r,\rho) = (\partial_\tau^l D_s^l p)(0, 0, ; r, \rho)$$

where

$$p(s, \tau; r, \rho) = b(r, \rho + \tau + \delta(r, r + s, \rho))a(r + s, \rho)$$

and

$$\delta(r, q, \rho) = \int_0^1 (\nabla_r \Phi_\lambda^\pm)((1 - t)r + tq), \rho)dt.$$

In particular, $d_l \in \mathcal{S}^{m_d - l} = \mathcal{S}^{-l}$ and

$$\begin{aligned} d_0(r,\rho) &= b(r, \rho + (\nabla_r \Phi_\lambda^\pm)(r,\rho))a(r,\rho), \\ d_1(r,\rho) &= (\partial_\rho b)(r, \rho + (\nabla_r \Phi_\lambda^\pm)(r,\rho))(D_r a)(r,\rho) \\ &+ \langle \partial_\rho^2 b(r, \rho + (\nabla_r \Phi_\lambda^\pm)(r,\rho)), \frac{1}{2}(\partial_r D_r \Phi_\lambda^\pm)(r,\rho)\rangle \; a(r,\rho). \end{aligned}$$

(27) implies that $d_1 \in \mathcal{S}^{-1-\nu}$ where $\nu = \max\{\nu_k, \nu_{a_1}^L\}$. Hence $d - d_0 \in \mathcal{S}^{-1-\nu}$.

Set $c(r,\rho) = \rho^2$. Then by Theorem 39 and Theorem 42, there exists $e \in \mathcal{S}^{m_e}$ with $m_e = 0$ such that

$$\begin{aligned} J_\lambda^\pm H_{0,\lambda} &= \chi_\lambda^\pm(D_r)J(\Phi_\lambda^\pm, a^\pm)(D_r^2) \\ &= J(\Phi_\lambda^\pm, e) \end{aligned}$$

and admits the asymptotic expansion

$$e = \sum_{l \geq 0} \frac{1}{l!} e_l,$$
$$e_l(r,\rho) = (\partial_\tau^l D_s^l q)(0, 0, ; r, \rho)$$

where

$$q(s,\tau;r,\rho)=a(r,\rho+\tau)\bar{c}(r+s+\gamma(r,\rho+\tau,\rho),\rho+\tau)$$

and

$$\gamma(r,\rho,\sigma)=\int_0^1(\nabla_\rho\Phi_\lambda^\pm)(r,(1-t)\rho+t\sigma)dt.$$

In particular, $e_l\in\mathcal{S}^{m_e-l}=\mathcal{S}^{-l}$ and

$$\begin{gathered}e_0(r,\rho)=a(r,\rho)\bar{c}(r+(\nabla_\rho\Phi_\lambda^\pm)(r,\rho),\rho),\\ e_1(r,\rho)=(\partial_\rho a)(r,\rho)(D_r\bar{c})(r+(\nabla_\rho\Phi_\lambda^\pm)(r,\rho),\rho)\\ +a(r,\rho)\big((D_r\partial_\rho\bar{c})(r+(\nabla_\rho\Phi_\lambda^\pm)(r,\rho),\rho)+\langle(\nabla_rD_r\bar{c})(r+(\nabla_\rho\Phi_\lambda^\pm)(r,\rho),\rho),\frac{1}{2}\nabla_\rho\partial_\rho\Phi_\lambda^\pm(r,\rho)\rangle\big).\end{gathered}$$

Since $c(r,\rho)=\rho^2$, $e_1=0$. Hence $e-e_1\in\mathcal{S}^{-2}$.

Now we have $T_\lambda^\pm=J(\Phi_\lambda^\pm,d-e)$ where $(d-e)-(d_0-e_0)\in\mathcal{S}^{-1-\nu}$ and

$$(d_0-e_0)(r,\rho)=\big(i^{-1}(\partial_ra_1^L)(r)(\rho+\nabla_r\Phi_\lambda^\pm(r,\rho))+R[\Phi_\lambda^\pm](r,\rho)\big)a(r,\rho),$$

where

$$R[\Phi_\lambda^\pm](r,\rho)=(1+a_1^L(r))|\rho+|\Phi_\lambda^\pm(r,\rho)|^2+\lambda k(r)-\rho^2.$$

As in Lemma 30, we chose $\Phi_\lambda^\pm$ so that $R[\Phi_\lambda^\pm](r,\rho)a(r,\rho)\in\mathcal{S}^{-1-\epsilon}$ with some $\epsilon>0$. Therefore $T_\lambda^\pm=J(\Phi_\lambda^\pm,d-e)$ with $d-e\in\mathcal{S}^{-1-\epsilon}$ and hence $\langle r\rangle^{\frac{1+\epsilon}{2}}T_\lambda^\pm\langle r\rangle^{\frac{1+\epsilon}{2}}$ is bounded. The operator $\langle r\rangle^{-\frac{1+\epsilon}{2}}$ is $H_{0,\lambda}$- and $H_{L,\lambda}$-smooth on any positive bounded interval disjoint from eigenvalues of $H_{L,\lambda}$. So the smooth perturbation theory of Kato yields the Lemma. □

Now we show that these wave operators are isometric on suitable subspaces.

Lemma 32.

$$\operatorname*{s-lim}_{t\to\pm\infty}((J^\pm)^*J^\pm-\psi(H_0))e^{-iH_0t}=0 \tag{30}$$

$$\operatorname*{s-lim}_{t\to\mp\infty}(J^\pm)^*J^\pm e^{-iH_0t}=0. \tag{31}$$

In particular, if $\Lambda\Subset\mathbb{R}_+$ and $\psi\in C_0^\infty(\mathbb{R})$ such that $\psi=1$ on Λ, then the wave operators $W^\pm(H_L,H_0;J^\pm)$ are isometric on the subspace $E_{H_0}(\Lambda)\mathcal{H}_f$ and $W^\pm(H_L,H_0;J^\mp)=0$.

Proof. Up to a compact term, $(J_\lambda^\pm)^*J_\lambda^\pm$ is a PDO $Q_\lambda^\pm$ with symbol

$$\eta^2(r)\psi^2(\rho^2)(\sigma^\pm)^2(r,\rho).$$

If $t\to\mp\infty$, then the stationary point $\rho=\frac{r}{2t}$ of the integral

$$(Q^\pm e^{-iH_{0,\lambda}t}u)(r)=\frac{1}{(2\pi)^{\frac{1}{2}}}\eta^2(r)^2\int_{\mathbb{R}}e^{ir\rho-i\rho^2t}\psi^2(\rho^2)(\sigma^\pm)^2(r,\rho)\hat{u}(\rho)d\rho.$$

does not belong to the support of the function $\sigma^\pm$. Therefore supposing $\hat{u}\in C_0^\infty(\mathbb{R})$ and integrating by parts, we estimate this integral by $C_N(1+|r|+|t|)^{-N}$ for an arbitrary N. This proves (31). We apply the same argument to the PDO with sumbol $\eta^2(r)\psi^2(\rho^2)(\sigma^\pm)^2(r,\rho)-\psi^2(\rho^2)$ to prove (30). □

From now on, fix Λ and ψ as in Lemma 32.

Lemma 33. *The wave operators $W^{\pm}(H_0, H_L; (J^{\pm})^*)$ are isometric on $E_{H_L}(\Lambda)\mathcal{H}_f$.*

Proof. By Lemma 32, $W^{\pm}(H_0, H_L; (J^{\mp})^*) = W^{\pm}(H_L, H_0; J^{\mp})^* = 0$. This implies

$$\lim_{t\to\pm\infty} \|J^{\mp *} e^{-iH_L t} u\| = 0,\ u \in E_{H_L}(\Lambda)\mathcal{H}_f. \tag{32}$$

Moreover, $J_\lambda^+(J_\lambda^+)^* + J_\lambda^-(J_\lambda^-)^* - \psi^2(H_{0,\lambda})$ and $\psi^2(H_{0,\lambda}) - \psi^2(H_{L,\lambda})$ are compact, and (32) implies that

$$\lim_{t\to\pm\infty} \|(J^{\pm})^* e^{-iH_L t} u\| = \|u\|,\ u \in E_{H_L}(\Lambda)\mathcal{H}_f.$$

This implies the Lemma. □

Lemma 34. *The wave operators $W^{\pm}(H_L, L_0 + D_r a_1^L D_r; J^*)$ are isometric on $P_{ac}(L_0)\mathcal{H}$.*

Proof. Use the local $L_0 + D_r a_1^L D_r$-smoothness of $1 - \chi$. □

Lemma 35. *The wave operators $W^{\pm}(L_0 + D_r a_1^L D_r, H_0; JJ^{\pm})$ are isometric on $E_\Lambda(H_0)\mathcal{H}_f^{\pm}$ and $W^{\pm}(L_0 + D_r a_1^L D_r, H_0; JJ^{\pm})\mathcal{H}_f^{\mp} = 0$*

Proof. It is enough to show that

$$\underset{t\to\pm\infty}{\text{s-lim}} [(JJ^{\pm})^* JJ^{\pm} - \psi(H_0)] e^{-iH_0 t} P_{\pm} = 0 \tag{33}$$

$$\underset{t\to\pm\infty}{\text{s-lim}} (JJ^{\pm})^* JJ^{\pm} e^{-iH_0 t} P_{\mp} = 0 \tag{34}$$

where $P_{\pm} = 0$ are projections onto the subspaces $\mathcal{H}_f^{\pm}$. Again up to a compact term, $(JJ^{\pm})^* JJ^{\pm} P_{\mp}$ is a PDO with symbol

$$\chi^2(r)\eta^2(r)\psi^2(\rho^2)(\sigma^{\pm})^2(r,\rho)1_{\mathbb{R}_{\mp}}(\rho) = 0.$$

This implies (34). Similarly, up to a compact term, $(JJ^{\pm})^* JJ^{\pm} P_{\pm}$ is a PDO with symbol

$$[(\chi^2(r)\eta^2(r)(\sigma^{\pm})^2(r,\rho) - 1]\psi^2(\rho^2)1_{\mathbb{R}_{\pm}}(\rho).$$

We apply the same argument as in Lemma 32 to prove (33). □

Combinig these results, we obtain the following theorem.

Theorem 36. *Suppose $\nu_{a_2} = \nu_{b_1} = \nu_{b_2} = \nu_V > 1$, $\nu_{a_3} = 1$ and a_1 can be separated into two parts as in (20) - (22). Suppose k is smooth long-range in the sense of Definition 4 and let the operators $J^{\pm}$ be defined by (23), (24), and (25) with $\Phi_\lambda^{\pm}$ satisfying the properties listed in Lemma 30 with k replaced by λk. We also assume that $\psi(\lambda) = 1$ on $\Lambda \Subset \mathbb{R}_+$, $\eta(r) = 0$ if $|r| < R$ for large enough R as is taken in Lemma 30. Then the wave operators $W^{\pm}(L, H_0; JJ^{\pm})$ and $W^{\pm}(H_0, L; (JJ^{\pm})^*)$ exist, are adjoint each other, are isometric on $E_\Lambda(H_0)\mathcal{H}_f^{\pm}$ and $E_\Lambda(L)P_{ac}(L)\mathcal{H}$, respectively, $W^{\pm}(L, H_0; JJ^{\pm})\mathcal{H}_f^{\mp} = 0$, and the asymptotic completeness*

$$W^{\pm}(L, H_0; JJ^{\pm})E_\Lambda(H_0)\mathcal{H}_f^{\pm} = E_\Lambda(L)P_{ac}(L)\mathcal{H}$$

holds.

A Pseudo-differential operators with oscillating symbols

In this appendix, we describe a class of pseudo-differential operators with oscillating symbols.

We recall the Hörmander classes $\mathcal{S}^m_{\rho,\delta}$ for $m \in \mathbb{R}, \rho > 0, \delta < 1$. We set $\mathcal{S}^m_{\rho,\delta} = \mathcal{S}^m_{\rho,\delta}(\mathbb{R}^d \times \mathbb{R}^d)$ consists of functions $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that, for all multi-indices α, β, there exist $C_{\alpha,\beta}$ such that

$$|(\partial_x^\alpha \partial_\xi^\beta a)(x,\xi)| \le C_{\alpha,\beta}(1+|x|)^{m-|\alpha|\rho+|\beta|\delta}$$

for all $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$. The best $C_{\alpha,\beta}$ are the semi-norms of the symbol a. We denote $\mathcal{S}^m = \mathcal{S}^m_{1,0}$. We say a symbol $a(x,\xi)$ is compactly supported in the variable ξ if there is a compact set $K \Subset \mathbb{R}^d$ such that

$$a(x,\xi) = 0$$

for all $x \in \mathbb{R}^d$ if $\xi \notin K$. We denote the pseudo-differential operator (PDO) with symbol $a(x,\xi)$ by $a(x,D)$

$$(a(x,D)u)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} a(x,\xi)\hat{u}(\xi)d\xi$$

where $\hat{u}$ is the Fourier transform of u

$$\hat{u}(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-i\langle x,\xi\rangle} u(x)dx.$$

The following is elementary.

Lemma 37. *Suppose that $a \in \mathcal{S}^m$ and a is compactly supported in the variable ξ. Then $a(x,D)\langle x\rangle^{-m}$ is bounded in the space $L^2(\mathbb{R}^d)$ and $a(x,D)\langle x\rangle^{-m'}$ is compact if $m' > m$.*

Now we define a class of symbols with oscillating factor. Let $\epsilon > 0, m \in \mathbb{R}$, $\Phi \in \mathcal{S}^{1-\epsilon}$, and $a \in \mathcal{S}^m$. We denote classes of symbols of the form

$$e^{i\Phi(x,\xi)}a(x,\xi)$$

by $C^m(\Phi)$. We denote the PDO with symbol $e^{i\Phi}a$ by $J(\Phi,a)$

$$J(\Phi,a) = (e^{i\Phi}a)(x,D).$$

Clearly $C^m(\Phi) \subset \mathcal{S}^m_{\epsilon,1-\epsilon}$ so that $C^m(\Phi)$ are good classes if $\epsilon > \frac{1}{2}$. On the other hand, the standard calculus fails for operators from these classes if $\epsilon \le \frac{1}{2}$. However as is shown in [20], $J(\Phi,a_1)J(\Phi,a_2)^*$ and $J(\Phi,a_1)^*J(\Phi,a_2)$ become usual PDO and admit asymptotic expansions.

Theorem 38. *Suppose that $\Phi \in \mathcal{S}^{1-\epsilon}$ with $\epsilon > 0$, and $a_j \in \mathcal{S}^{m_j}$ for $j = 1,2$ and some numbers m_j. Suppose that a_j are compactly supported in the variable ξ. Then the following holds.*

(i). $G = J(\Phi,a_1)J(\Phi,a_2)^*$ is a PDO with symbol $g \in \mathcal{S}^m$ for $m = m_1 + m_2$ and $g(x,\xi)$ admits the asymptotic expansion

$$g = \sum_{|\alpha|\ge 0} \frac{1}{\alpha!} g_\alpha,$$
$$g_\alpha(x,\xi) = \partial_\xi^\alpha(e^{i\Phi(x,\xi)}a_1(x,\xi)D_x^\alpha(e^{-i\Phi(x,\xi)}\bar{a}_2(x,\xi)));$$

in particular, $g_\alpha \in \mathcal{S}^{m-|\alpha|\epsilon}$.

(ii). $H = J(\Phi, a_2)^* J(\Phi, a_1)$ *is a PDO with symbol* $h \in \mathcal{S}^m$ *for* $m = m_1 + m_2$ *and* $h(x,\xi)$ *admits the asymptotic expansion*

$$h = \sum_{|\alpha|\geq 0} \frac{1}{\alpha!} h_\alpha,$$
$$h_\alpha(x,\xi) = D_x^\alpha (e^{i\Phi(x,\xi)} a_1(x,\xi) \partial_\xi^\alpha (e^{-i\Phi(x,\xi)} \bar{a}_2(x,\xi)));$$

in particular, $h_\alpha \in \mathcal{S}^{m-|\alpha|\epsilon}$.

(iii). $J(\Phi, a_1)$ *is bounded in the space* $L^2(\mathbb{R}^d)$ *if* $m_1 = 0$ *and it is compact if* $m_1 < 0$.

(iv). *Suppose* $m_1 = m_2 = 0$. *Denote by* A *the PDO with symbol*

$$a(x,\xi) = a_1(x,\xi)\bar{a}_2(x,\xi) \in \mathcal{S}^0.$$

Then $J(\Phi, a_1)J(\Phi, a_2)^* - A$ *and* $J(\Phi, a_2)^* J(\Phi, a_1) - A$ *are compact in* $L^2(\mathbb{R}^d)$.

For the proof of Theorem 38, we refer Yafaev [20].

Next we consider the product of a PDO with oscillating symbol and a usual pseudo-differential operator. The situation is different whether the pseudo-differential operator is on the left and on the right.

Theorem 39. *Suppose that* $\Phi \in \mathcal{S}^{1-\epsilon}$, $a \in \mathcal{S}^{m_a}$, *and* $b \in \mathcal{S}^{m_b}$ *for* $\epsilon > 0$ *and some* $m_a, m_b \in \mathbb{R}$. *Suppose* a *and* b *are compactly supported in the variable* ξ. *Then there exists a symbol* $d \in \mathcal{S}^{m_d}$ *for* $m_d = m_a + m_b$ *such that* d *is compactly supported in the variable* ξ,

$$b(x,D)J(\Phi,a) = J(\Phi,d),$$

and admits the asymptotic expansion

$$d = \sum_{|\alpha|\geq 0} \frac{1}{\alpha!} d_\alpha,$$
$$d_\alpha(x,\eta) = (\partial_\zeta^\alpha D_z^\alpha p)(0,0,;x,\eta)$$

where

$$p(z,\zeta;x,\eta) = b(x, \eta + \zeta + r(x, x+z, \eta)) a(x+z, \eta)$$

and

$$r(x,y,\eta) = \int_0^1 (\nabla_x \Phi)((1-\tau)x + \tau y), \eta) d\tau.$$

In particular, $d_\alpha \in \mathcal{S}^{m_d - |\alpha|}$ *and*

$$d_0(x,\eta) = b(a, \eta + (\nabla_x\Phi)(x,\eta)) a(x,\eta),$$
$$d_\alpha(x,\eta) = (\partial_\eta^\alpha b)(a, \eta + (\nabla_x\Phi)(x,\eta))(D_x^\alpha a)(x,\eta)$$
$$+ \langle \nabla_\eta \partial_\eta^\alpha b(a, \eta + (\nabla_x \Phi)(x,\eta)), \frac{1}{2}(\nabla_x D_x^\alpha \Phi)(x,\eta)\rangle a(x,\eta)$$

if $|\alpha| = 1$.

Proof. We compute

$$
\begin{aligned}
&(b(x,D)J(\Phi,a)u)(x)\\
=&(2\pi)^{-\frac{3n}{2}}\int e^{i\langle x,\xi\rangle - i\langle y,\xi\rangle - +i\langle y,\eta\rangle + i\Phi(y,\eta)} b(x,\xi)a(y,\eta)\hat{u}(\eta)d\eta dy d\xi\\
=&(2\pi)^{-\frac{3n}{2}}\int e^{i\langle x,\eta\rangle + i\Phi(x,\eta)}\hat{u}(\eta)\big(\int e^{i\langle x-y,\xi-\eta\rangle + i(\Phi(y,\eta)-\Phi(x,\eta))} b(x,\xi)a(y,\eta)dyd\xi\big)d\eta\\
=&(2\pi)^{-\frac{n}{2}}\int e^{i\langle x,\eta\rangle + i\Phi(x,\eta)}\hat{u}(\eta)d(x,\eta)d\eta
\end{aligned}
$$

where

$$
d(x,\eta) = (2\pi)^{-n}\int e^{i\langle x-y,\xi-\eta\rangle + i(\Phi(y,\eta)-\Phi(x,\eta))} b(x,\xi)a(y,\eta)dyd\xi.
$$

We set

$$
r(x,y,\eta) = \int_0^1 (\nabla_x\Phi)((1-\tau)x+\tau y,\eta)d\tau.
$$

Then

$$
\Phi(y,\eta) - \Phi(x,\eta) = \langle y-x, r(x,y,\eta)\rangle.
$$

By changing variables, we compute

$$
\begin{aligned}
&d(x,\eta)\\
=&(2\pi)^{-n}\int e^{i\langle x-y,\xi-\eta-r(x,y,\eta)\rangle} b(x,\xi)a(y,\eta)dyd\xi\\
=&(2\pi)^{-n}\int e^{i\langle x-y,\tilde{\xi}-\eta\rangle} b(x,\tilde{\xi}+r(x,y,\eta))a(y,\eta)dyd\xi\\
=&(2\pi)^{-n}\int e^{-i\langle z,\zeta\rangle} b(x,\eta+\zeta+r(x,x+z,\eta))a(x+z,\eta)dyd\xi.
\end{aligned}
$$

Set

$$
p(z,\zeta;x,\eta) = b(x,\eta+\zeta+r(x,x+z,\eta))a(x+z,\eta).
$$

Then by Taylor's expansion formula, we obtain the folowing:

$$
d(x,\eta) = \sum_{0\le|\alpha|\le N-1}\frac{1}{\alpha!}(\partial_\zeta^\alpha D_\zeta^\alpha p)(0,0;x,\eta) + p^{(N)}(x,\eta)
$$

where

$$
p^{(N)}(x,\eta) = (2\pi)^{-n}N\sum_{|\alpha|=N}\frac{1}{\alpha!}\int_0^1(1-t)^{N-1}\int\int(\partial_z^\alpha p)(tz,\zeta;x.\eta)z^\alpha e^{-i\langle z,\zeta\rangle}dzd\zeta dt.
$$

Set

$$
R^{(\alpha)}(x,\eta;t) = \int\int(\partial_z^\alpha D_\zeta^\alpha p)(tz,\zeta;x.\eta)e^{-i\langle z,\zeta\rangle}dzd\zeta.
$$

Now it is enough to show that $R^{(\alpha)}\in S^{m_d-|\alpha|}$ and the seminorms are bounded uniformly with respect to the variable t. This obeys from the following two elementary lemmas.

Lemma 40. *Fix $C > 0$. If $|z| \geq C|x|$, then for any n,*

$$|\int (\partial_z^\alpha D_\zeta^\alpha p)(tz, \zeta; x.\eta) e^{-i\langle z,\zeta\rangle} d\zeta| \leq C\langle z\rangle^{-n}.$$

Lemma 41. *There exists $C > 0$ such that*

$$|\int \int_{|z|\leq C|x|} (\partial_z^\alpha D_\zeta^\alpha p)(tz, \zeta; x.\eta) e^{-i\langle z,\zeta\rangle} dz d\zeta| \leq C\langle x\rangle^{m_d - |\alpha|}.$$

By integrating by parts, we can show these lemmas. □

Theorem 42. *Suppose that $\Phi \in \mathcal{S}^{1-\epsilon}$, $a \in \mathcal{S}^{m_a}$, and $c \in \mathcal{S}^{m_c}$ for $\epsilon > 0$ and some $m_a, m_c \in \mathbb{R}$. Suppose a is compactly supported in the variable ξ. Then there exists a symbol $e \in \mathcal{S}^{m_e}$ for $m_e = m_a + m_c$ such that*

$$J(\Phi, a) c(x, D)^* = J(\Phi, e),$$

and admits the asymptotic expansion

$$e = \sum_{|\alpha|\geq 0} \frac{1}{\alpha!} e_\alpha,$$
$$e_\alpha(x, \eta) = (\partial_\zeta^\alpha D_z^\alpha q)(0, 0, ; x, \eta)$$

where

$$q(z, \zeta; x, \eta) = a(x, \eta + \zeta)\bar{c}(x + z + s(x, \eta + \zeta, \eta), \eta + \zeta)$$

and

$$s(x, \zeta, \eta) = \int_0^1 (\nabla_\xi \Phi)(x, (1 - \tau)\eta + \tau\xi) d\tau.$$

In particular, $e_\alpha \in \mathcal{S}^{m_e - |\alpha|}$ and

$$e_0(x, \eta) = a(x, \eta)\bar{c}(x + (\nabla_\eta \Phi)(x, \eta), \eta),$$
$$e_\alpha(x, \eta) = (\partial_\eta^\alpha a)(x, \eta)(D_x^\alpha \bar{c})(x + (\nabla_\eta \Phi)(x, \eta), \eta)$$
$$+a(x, \eta)\big((D_x^\alpha \partial_\eta \bar{c}(x + (\nabla_\eta \Phi)(x, \eta), \eta) + \langle(\nabla_x D_x^\alpha \bar{c})(x + (\nabla_\eta \Phi)(x, \eta), \eta), \frac{1}{2}\nabla_\xi \partial_\xi^\alpha \Phi(x, \eta)\rangle\big)$$

if $|\alpha| = 1$.

Proof is similar to Theorem 39.

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